Solving systems of linear equations by elimination

Suppose we are looking for real numbers x, y that solve both of the following equations.

$$① \quad 3x + 2y = 2$$

$$② \quad 5x + 6y = 3$$

We can start by solving ② for y in terms of x, to get

$$6y = 3 - 5x$$

$$(*) \quad y = \frac{3-5x}{6}.$$

We can then substitute $(*)$ in ① to get

$$3x + 2\left(\frac{3-5x}{6}\right) = 2$$

$$\frac{4}{3}x + 1 = 2$$

$$(**) \quad x = \frac{3}{4}.$$

Finally, we substitute $(**)$ in $(*)$ to get

$$y = \frac{3 - 5\left(\frac{3}{4}\right)}{6} = -\frac{1}{8}.$$

Thus $x = \frac{3}{4}$, $y = -\frac{[illegible]}{8}$.

Substituting these values in our original equation confirms that this is a solution.

Now consider a more complex version of this question:

Ex:

$$
\begin{aligned}
3x + 2y + z &= 2 \\
x - 4y + z &= 0 \\
5x - 5y - z &= 3
\end{aligned}
$$

The same techniques that we used above will work on this problem too, but the

will work on [illegible]
process becomes time-consuming.

Such problems, of the form

Find real numbers $x_1, x_2, \dots, x_n$ which solve all of the equations

$$m \text{ equations} \left\{ \begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ \underbrace{a_{m1}x_1 + \dots + a_{mn}x_n}_{\text{unknowns}} &= b_m \end{aligned} \right.$$

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are known as "Systems of linear equations"

In real world applications, one often encounters systems of linear equations with millions of unknowns & equations.

Obviously, we need a more systematic approach.

Elimination:

We start by introducing the method of elimination. Consider

Ex: ① $3x + 2y = 1$

② $2x + 2y = 5$.

We want to 'eliminate' y from ②.

Let's subtract the left-hand side of ① from the left-hand side of ②, and the right-hand side of ① from the right-hand side of ②. We get

$$②' : \quad 2x - 3x + 2y - 2y = 5 - 1.$$

or

$$-x + 0 \cdot y = 4.$$

This equation is still satisfied by any solution (x, y) of the original equation, since we subtracted the same quantity on either side.

But it's easier to solve:

$$0y = 4 \longrightarrow \quad x = -4.$$

$-x + 5y$

Now we can go back & solve for y:

$$3x + 2y = 1$$
$$3(-4) + 2y = 1$$
$$y = \frac{13}{2}.$$

Elimination is the process of 'eliminating' variables in equations by subtracting multiples of other equations. Here's another example:

Ex: ① $x + 2y + z = 0$

② $2x + y + z = 1$

③ $x + y + 2z = 0$

subtract ① from ③ to eliminate x:

① $x + 2y + z = 0$

② $2x + 3y + z = 1$

③′ = ③ − ① $0x - y + z = 0$

subtract 2×① from ② to eliminate x:

① $x + 2y + z = 0$

$$\text{(1)}\quad x + 2y + z = 0$$

$$\text{(2')} = \text{(2)} - 2\text{(1)} \quad 0x - y - z = 1$$

$$\text{(3')} \quad 0x - y + z = 0$$

subtract (2) from (3') to eliminate y:

$$\text{(1)} \quad x + 2y + z = 0$$

$$\text{(2')} \quad 0x - y - z = 1$$

$$\text{(3'')} = \text{(3')} - \text{(2)} \quad 0x + 0y + 2z = -1.$$

Now we have only one unknown left in (3'').

We can solve: $z = -\frac{1}{2}$.

Substituting in (2'), we have one unknown:

$$-y - \left(-\frac{1}{2}\right) = 1$$

$$y = -\frac{1}{2}.$$

Finally, substituting in 1:

$$1 \qquad x + 2\left(-\tfrac{1}{2}\right) + \left(-\tfrac{1}{2}\right) = 0$$

$$x = \frac{3}{2}.$$

Notice how we kept eliminating until we had only one unknown left in the last equation, two in the second-last, etc.

Sometimes, to carry out this strategy,

we need to swap equations:

Ex: (1) $0 \cdot x + 2y + z = 2$

(2) $x + y + z = 1$

(3) $x - y + z = 4$.

If we try to proceed as before, we won't be able to eliminate x from (2) or (3) by subtracting multiples of (1), since (1) has $\underline{0}$ as coefficient of x.

swap (1) & (3):

So we study ...

$$\begin{aligned}
1' = 3 \quad & x - y + z = 4 \\
2 \quad & x + y + z = 1 \\
3' = 1 \quad & 0x + 2y + z = 2.
\end{aligned}$$

Now we can solve as in the previous example.

$\boxed{\text{Definition:}}$ the <u>solution set</u> of a system of linear equations is the set of all values of $x_1, \ldots, x_n$ which solve all the given equations.

all the given eq

Ex: $$\begin{aligned}3x+2y&=2\\2x-y&=0\end{aligned}$$ Solution set $= \left(\frac{2}{7}, \frac{4}{7}\right)$

Ex: $$\begin{aligned}3x+2y&=2\\6x+4y&=4\end{aligned}$$ solution set $= \left\{\left(a, \frac{2-3a}{2}\right)\right\}$ where a is any real number

Ex: $$\begin{aligned}3x+2y&=2\\6x+4y&=3\end{aligned}$$ solution set $= \{\text{nothing}\}$

These 3 examples illustrate the 3 possible types of solution sets:

$\boxed{\text{Theorem:}}$ A system of equations can have either:

1) a unique solution.
2) ∞ many solutions.
3) no solutions.

Equation operations: ways of modifying a system of linear equations by

1) Subtracting a multiple of one equation from another.

... equations.

2) Swapping 2 equ...

3) Multiplying an equation by a nonzero constant.

We have been using these above. The point of equation operations is that they preserve the solution set, i.e.

Theorem: If a system of equations S1 is obtained from a different system

S2 by a sequence of
equation operations, then S1 & S2
have the same solution sets.